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LETTER TO THE EDITOR

Fourth-order differential equations for numerator polynomials

A Ronveaux

Physique Mathématique, Facultés Universitaires Notre-Dame de la Paix, 5000 Namur, Belgium

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Abstract. We give explicitly the fourth-order differential equation satisfied by the numerator polynomials (associated polynomials) of the classical orthogonal polynomials. The coefficients of the differential equation are at most a quadratic combination of the polynomials σ and τ (and their derivatives) defined via the relation $(\sigma\rho)' = \tau\rho$. This differential equation is therefore valid for the Jacobi, Laguerre, Hermite and Bessel associated polynomials.

The classical orthogonal polynomials p_n (Jacobi, Laguerre, Hermite) are characterised by a weight $\rho = \rho(x)$, positive in the interval (a, b), solution of a first-order differential equation:

$$(\sigma\rho)' = \tau\rho \qquad \int_a^b \rho x^k \, \mathrm{d}x < \infty \quad \forall k \tag{1}$$

and are solutions of a second-order differential equation of the hypergeometric type:

$$\sigma p_n'' + \tau p_n' + \lambda_n p_n = 0 \tag{2}$$

where σ is a polynomial in x of degree 2, 1, or 0; τ is a polynomial of degree 1 and λ_n is a constant.

In this letter, we give the differential equation satisfied by the so-called first associated polynomials (or numerator polynomials) $p_{n-1}^{(1)}(x) = r$ defined by

$$r = p_{n-1}^{(1)}(x) = \int_a^b \frac{p_n(s) - p_n(x)}{s - x} \rho(s) \, \mathrm{d}s \tag{3}$$

where $p_n(x)$ is any classical orthogonal polynomial (including the Bessel polynomials).

These polynomials play an important role in the theory of orthogonal polynomials (Szegö 1939, Chihara 1978, Askey and Wimp 1984, Wimp 1985, Grosjean 1986), in the asymptotic theory of the solution of the second kind of equation (2) (Nikiforov and Ouvarov 1983), and in the study of the Laguerre–Hahn class of orthogonal polynomials (Magnus 1983, Dzoumba 1985). The numerator polynomials $p_{(n-1)}^{(1)}(x)$ corresponding to classical p_n (excluding the Tchebichef family) belong to the Laguerre–Hahn class and are therefore solutions of a fourth-order differential equation which is only obtained in the Jacobi case (Wimp 1986) after extensive mathematical developments and heavy MACSYMA manipulation and in the Legendre case by hand computation (Grosjean 1985).

The recurrence relation:

$$x t_n = \alpha_n t_{n+1} + \beta_n t_n + \gamma_n t_{n-1} \tag{4}$$

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is satisfied by the family of polynomials $p_n(x)$ and also by the function $q_n(x)=q_n$ defined as

$$q_n(x)=\int_a^b \frac{p_n(s)\rho(s)\,\mathrm{d}s}{s-x} \qquad x\notin[a,b]. \tag{5}$$

From definition (3) we obtain immediately the well known relation

$$r=q_n-p_nq_0/a_0 \tag{6}$$

where a_n is the leading coefficient of $p_n(x)$.

A differential equation for r can be obtained in two steps from the basic relation (6) using a fundamental result (Nikiforov and Ouvarov 1983), based on the hypergeometric character of the differential equation (2), and by saying that

$$Q_n=Q_n(x)=q_n(x)/\rho \tag{7}$$

is also a solution of equation (2). This result in the Legendre and Jacobi case is already known (Hobson 1931, Erdely *et al* 1953, Szegö 1939).

The differential equation

$$\sigma\left(\frac{r}{\rho}+\frac{p_nQ_0}{a_0}\right)''+\tau\left(\frac{r}{\rho}+\frac{p_nQ_0}{a_0}\right)'+\lambda_n\left(\frac{r}{\rho}+\frac{p_nQ_0}{a_0}\right)=0 \tag{8}$$

can be developed in the following way:

$$\sigma\left(\frac{r}{\rho}\right)''+\tau\left(\frac{r}{\rho}\right)'+\lambda_n\left(\frac{r}{\rho}\right)+\frac{1}{a_0}(\tau p_nQ_0'+2\sigma p_n'Q_0'+\sigma p_nQ_0'')=0. \tag{9}$$

The differential equation satisfied by Q_0 reduces to:

$$\sigma Q_0''+\tau Q_0'=0 \qquad \lambda_0=0 \tag{10}$$

and therefore (9) becomes

$$\sigma\left(\frac{r}{\rho}\right)''+\tau\left(\frac{r}{\rho}\right)'+\lambda_n\left(\frac{r}{\rho}\right)+\frac{2\sigma}{a_0}p_n'Q_0'=0. \tag{11}$$

$Q_n(x)$, being a solution of (2), can also be written as

$$Q_n=p_n\int\frac{\mathrm{d}x}{p_n^2}\exp\left(-\int\frac{\tau}{\sigma}\,\mathrm{d}x\right) \tag{12}$$

and using the weight differential equation (1), equation (12) becomes

$$Q_n=p_n\int\frac{\mathrm{d}x}{\sigma\rho p_n^2}. \tag{13}$$

For $n=0$ (Nikiforov and Ouvarov 1983), we obtain therefore (in accordance with (10))

$$\sigma\rho Q_0'=\text{constant}=K. \tag{14}$$

Using this result in (11) we deduce the second-order non-homogeneous differential equation satisfied by r:

$$\sigma\rho\left(\frac{r}{\rho}\right)''+\tau\rho\left(\frac{r}{\rho}\right)'+\lambda_nr=\bar{K}p_n' \qquad \bar{K}=(\sigma''-2\tau')\int_a^b\rho(x)\,\mathrm{d}x. \tag{15}$$

After simplification, using (1) written in the following way:

$$\frac{\rho'}{\rho}=\frac{\tau-\sigma'}{\sigma} \tag{16}$$

and

$$\left(\frac{\rho'}{\rho^2}\right)'=\frac{1}{\rho\sigma^2}[\sigma(\tau'-\sigma'')-\tau(\tau-\sigma')] \tag{17}$$

this first step therefore links r and p_n' in the following way:

$$\sigma r''+r'(2\sigma'-\tau)+r(\lambda_n+\sigma''-\tau')=Kp_n' \tag{18}$$

and this result generalises to all classical polynomials a relation given in the Jacobi and Gegenbauer case using heavy algebraic computation (Grosjean 1986).

The second step is now almost trivial, by computing the derivative of (2) (in which λ_n is a constant):

$$\sigma p_n'''+p_n''(\sigma'+\tau)+p_n'(\lambda_n+\tau')=0. \tag{19}$$

We are therefore forced to compute the second derivative of (18) in order to eliminate p_n' using (19).

After elementary computation we easily obtain the fourth-order differential equation satisfied by $r=p_{n-1}^{(1)}(x)$:

$$\sigma^2 r^{(\mathrm{IV})}+5\sigma\sigma' r'''+r''[\sigma(6\sigma''-2\tau'+2\lambda_n)+\tau(2\sigma'-\tau)+3\sigma'^2]+3r'[\sigma''(\sigma'+\tau)-\tau\tau'+\sigma'\lambda_n]$$
$$+r(\lambda_n+\tau')(\lambda_n+\sigma''-\tau')=0. \tag{20}$$

In the Hermite case ($p_n=H_n$, $\sigma=1$, $\tau=-2x$, $\lambda_n=2n$) this reduces to:

$$r^{(\mathrm{IV})}+r''(2\lambda_n+4-4x^2)-12xr'+r(\lambda_n^2-4)=0. \tag{21}$$

In the Laguerre case ($p_n=L_n^\alpha$, $\sigma=x$, $\tau=1+\alpha-x$, $\lambda_n=n$) equation (20) reduces to

$$x^2r^{(\mathrm{IV})}+5xr'''+r''[2x(\lambda_n+1)+4-(\alpha-x)^2]+3r'(\lambda_n+1+\alpha-x)+r(\lambda_n^2-1)=0. \tag{22}$$

In the Legendre case ($p_n=P_n$, $\sigma=1-x^2$, $\tau=-2x$, $\lambda_n=n(n+1)$) equation (20) reduces to

$$(1-x^2)^2r^{(\mathrm{IV})}-10x(1-x^2)r'''+2r''[(1-x^2)(\lambda_n-4)+8x^2]$$
$$+6xr'(2-\lambda_n)+\lambda_n(\lambda_n-2)r=0. \tag{23}$$

In the Bessel case ($p_n=y_n$, $\sigma=x^2$, $\tau=2x+2$, $\lambda_n=-n(n+1)$) equation (20) reduces to

$$x^4r^{(\mathrm{IV})}+10x^3r'''+2r''[x^2(\lambda_n+12)-2]+2xr'(\lambda_n+2)+r\lambda_n(\lambda_n+2)=0. \tag{24}$$

From $\lambda_n=-n[\tau'+\frac{1}{2}(n-1)\sigma'']$, we check immediately that $\lambda_n+\tau'$ is zero when $n=1$, indicating that $p_0^{(1)}$ is a solution of the equation.

The polynomials $p_{n-1}^{(1)}$ can be obtained recursively in an easy way, computing the integral given in (3) for $n=1, 2$ and the following by the recurrence relation (4) where α_n, β_n and γ_n are of course well known as functions of n. We must take care, however, that following (6) r is also a solution of (4) but of degree $(n-1)$ and the computation of $p_n^{(1)}(x)$ $(n>1)$ is therefore obtained by relation (4) with $\alpha_{n+1}, \beta_{n+1}, \gamma_{n+1}$.

This recurrence method allows us to check the differential equation because closed formulae for polynomials $p_{n-1}^{(1)}(x)$ are complicated or unknown, except in the Legendre case where a nice formula is known (Hobson 1931)

$$p_n^{(1)} = \sum_{k=0}^{n} \frac{P_k(x)P_{n-k}(x)}{k+1}. \tag{25}$$

It is also well known that the Tchebichef polynomials of the second kind $U_n(x)$ are the numerator polynomials for $U_n(x)$ itself and also for $T_n(x)$ the Tchebichef polynomials of the first kind.

This does not imply that equation (20) coincides in these cases with the second derivative of the differential equation satisfied by $U_n(x)$. It is only an appropriate linear combination of the three equations: the original one (satisfied by U_n or T_n), the first derivative of the original one, and the second derivative of the original one, which fits the fourth-order equation.

In the general case, starting with any family of orthogonal polynomials, their first associated polynomials also satisfy a fourth-order differential but the task of obtaining it is truly formidable because the hypergeometric equation is lost when we leave the classical orthogonal polynomials.

The value of Q_0' given in (14) can also be used to obtain a third-order linear differential equation for the inverse w_1 of the weight ρ_1 ($w_1 = 1/\rho_1$) of the associated polynomials of the classical (or the superclassical $\bar{\rho} = \Pi\rho_{\text{class}}$, Π polynomials (Ronveaux 1987)).

The representation of the weight ρ_1 (Maroni 1982) can be written in the following way ($\Pi = 1$):

$$\rho\rho_1(Q_0^2 + A) = 1 \qquad A = \text{constant}. \tag{26}$$

By logarithmic differentiation, use of (14) and elimination of $Q_0^2 + A$ we easily obtain

$$2KQ_0\rho_1 + \tau - \sigma' + \sigma\rho_1'/\rho_1 = 0. \tag{27}$$

This relation becomes linear in $w_1 = 1/\rho_1$ and can be written as

$$2KQ_0 = \sigma w_1' + (\sigma' - \tau)w_1. \tag{28}$$

We need again to differentiate twice in order to eliminate $Q_0' = (K/\sigma)w$ ($w = 1/\rho$) from (1) written as

$$\sigma w' + (\tau - \sigma')w = 0. \tag{29}$$

The result is

$$\sigma^2 w_1''' + 3\sigma\sigma' w_1'' + w_1'[\sigma(3\sigma'' - 2\tau') + \tau(2\sigma' - \tau)] + w_1[\tau(\sigma'' - \tau')] = 0 \tag{30}$$

which extends a result given by Grosjean (1985).

Note added. We realised after sending this paper for publication that equations (20) and (30) also appear, without proofs, in unpublished work by Grosjean.

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